TEMPERATURE FIELD IN A SOLID WITH A NONUNIFORMLY HEATED SURFACE

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In problems associated with aerodynamic heating, the process of heat propagation in a solid is usually considered one-dimensional (more exactly, as quasi-one-dimensional) by assuming a parametric dependence on the coordinate along the heated surface. However, measurements of thermal fluxes [1] show that the influence of longitudinal heat conduction can be substantial even for comparatively short heating times. In this connection it is expedient to clarify and estimate the deflection of the temperature distribution from the one-dimensional in certain simple problems taking account of the non-uniform surface heating. A model problem of body heating in the neighborhood of a critical point and the problem of body heating behind a thermal front moving along its surface are considered below.

1. Aerodynamic heating of a body in the neighborhood of a forward critical point. Let us consider the following problem (plane $\nu = 1$ and axisymmetric $\nu = 2$) on the heating of a semi-infinite body $y > 0$ which has the same temperature T_∞ everywhere at the initial instant with a given normal derivative on the boundary

$$\frac{1}{a^2}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{\nu - 1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial y^2}$$

$$t = 0, \qquad y = \infty, \qquad T = T_\infty \tag{1.1}$$

$$y = 0, \qquad \frac{\partial T}{\partial y} = -T_\infty k\,(\beta + \alpha e^{-k^2 r^2}), \qquad r = 0, \qquad \frac{\partial T}{\partial r} = 0$$

Problem (1.1) may be considered as a model to describe the process of heat propagation at the nose section of a blunt body subjected to aerodynamic heating. Let us transform to nondimensional quantities by means of Formulas

$$r' = kr, \qquad y' = ky, \qquad t' = k^2 a^2 t, \qquad \theta = \frac{T - T_\infty}{T_\infty}$$

Problem (1.1) is formulated in nondimensional quantities as

$$\frac{\partial\theta}{\partial t'} = \frac{\partial^2\theta}{\partial r'^2} + \frac{\nu - 1}{r'}\frac{\partial\theta}{\partial r'} + \frac{\partial^2\theta}{\partial y'^2}$$

$$t' = 0, \qquad y = \infty, \qquad \theta = 0 \tag{1.2}$$

$$y' = 0, \qquad \frac{\partial\theta}{\partial y'} = -(\beta + \alpha e^{-r'^2}), \qquad r' = 0, \qquad \frac{\partial\theta}{\partial r'} = 0$$

For simplicity of writing, the primes are omitted below. Hence, the solution of problem (1.2) depends on the two nondimensional parameters α and β, and, because of linearity of the problem, it can be written

$$\theta = \theta(t, r, y; \alpha, \beta) = \theta_1(t, r, y; \alpha) + \theta_2(t, y; \beta)$$

For the function θ_2 satisfying the boundary condition

$$\partial\theta_2 / \partial y = -\beta \quad \text{for} \quad y \to 0,$$

we have

$$\theta_2(t, y; \beta) = \beta \int_0^t \exp\left(-\frac{y^2}{4\tau}\right) \frac{d\tau}{\sqrt{\pi\tau}}$$

Let us seek the function θ_1 as the Fourier integral

$$\theta_1(t, r, y; \alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty \vartheta(t, s, y) \cos sr \, ds \qquad \text{for } \nu = 1$$

or as the Fourier-Bessel integral

$$\theta_1(t, r, y; \alpha) = \int_0^\infty \vartheta(t, s, y) J_0(sr) \, s ds \qquad \text{for } \nu = 2$$

Here $J_0(x)$ is the zero order Bessel function of the first kind. Then we will have the following problem for ϑ :

$$\frac{\partial\vartheta}{\partial t} = \frac{\partial^2\vartheta}{\partial y^2} - s^2\vartheta$$
$$t = 0, \qquad \vartheta = 0; \qquad y = 0, \qquad \frac{\partial\vartheta}{\partial y} = \frac{1}{(\sqrt{2})^\nu} \alpha \exp\left(-\frac{s^2}{4}\right) \tag{1.3}$$

Problem (1.3) may be solved by an operational method. We obtain for $\vartheta(t, s, y)$

$$\vartheta(t, s, y) = \frac{\alpha}{(\sqrt{2})^\nu} \exp\left(-\frac{s^2}{4}\right) \int_0^t \exp\left(-\frac{y^2}{4\tau}\right) e^{-s^2\tau} \frac{d\tau}{\sqrt{\pi\tau}}$$

Using the formulas of the inverse Fourier and Fourier-Bessel transforms for the function $\exp(-ks^2)$ (see [2] pp.216 and 585) we find θ_1

$$\theta_1 = \alpha \int_0^t \exp\left(-\frac{y^2}{4\tau} - \frac{r^2}{4\tau + 1}\right) \frac{d\tau}{\sqrt{\pi\tau (4\tau + 1)^\nu}} \tag{1.4}$$

Hence, the problem (1.1) is solved. Let us compare the obtained solution with the quasi-one-dimensional solution. The latter has the form

$$\theta_0 = (\beta + \alpha e^{-r^2}) \int_0^t \exp\left(-\frac{y^2}{4\tau}\right) \frac{d\tau}{\sqrt{\pi\tau}} = (\beta + \alpha e^{-r^2}) \, 2\sqrt{t} \, \text{ierfc} \, \frac{y}{2\sqrt{t}} \tag{1.5}$$

Here (see [3])

$$\text{ierfc}\, x = \int_x^\infty \text{erfc}\, x dx$$

As the measure of the deviation of the quasi-one-dimensional solution from the exact solution, let us take the relative difference in temperatures given by both solutions at the point $r = y = 0$. The difference thus determined we shall denote by $\Delta\theta_*$ and it equals

$$\Delta\theta_* = \frac{\theta_0(t,0,0) - \theta(t,0,0)}{\theta_0(t,0,0)} = \frac{\alpha}{\alpha+\beta}\,\frac{1}{2\sqrt{t}}\int_0^t \left[\frac{1}{(4\tau+1)^{1/2\nu}} - 1\right]\frac{d\tau}{\sqrt{\tau}}$$

For definiteness, let $\nu = 2$. Then

$$\Delta\theta_* = \frac{\alpha}{\alpha+\beta}\left(1 - \frac{\tan^{-1} 2\sqrt{t}}{2\sqrt{t}}\right) \tag{1.6}$$

For small t the quantity $\Delta\theta_*$ grows linearly with time

$$\Delta\theta_* = \frac{\alpha}{\alpha+\beta}\,\frac{4}{3}\,t \tag{1.7}$$

In order that the obtained result might be applied to steady problems, let us express $\Delta\theta_*$ in terms of the nondimensional characteristic thickness δ of the thermal layer. We define δ as the distance from the surface at which the temperature gradient, computed by the quasi-one-dimensional solution, is 0.5% of the same quantity on the surface. Since

$$\frac{\partial\theta_0}{\partial y} = -\left(\beta + \alpha e^{-r^2}\right)\operatorname{erfc}\frac{y}{2\sqrt{t}}$$

then $\delta = 4\sqrt{t}$. Hence, we obtain from (1.6)

$$\Delta\theta_* = \frac{\alpha}{\alpha+\beta}\left(1 - \frac{2}{\delta}\tan^{-1}\frac{\delta}{2}\right) \tag{1.8}$$

For small δ

$$\Delta\theta_* = \frac{\alpha}{\alpha+\beta}\,\frac{\delta^2}{12} \tag{1.9}$$

From formulas obtained it follows that the deviation of the temperature distribution from the one-dimensional will be noticeable if the thickness of the thermal layer exceeds the characteristic length of the change in the heat flux at the boundary.

2. Heating of a wall behind a moving thermal front was considered by Tirskii [4] in the steady-state problem (in a moving coordinate system) of heating of a half-space behind a moving compression shock by neglecting the flowing of the heat along the body surface because of heat conduction (quasi-one-dimensional solution). The same problem, not absolutely steady-state, is considered below, however, only for a solid without the assumption of quasi-one-dimensionality of the process. It is assumed that the heat flux on the surface of the body is known.

Let a thermal front move with velocity V along the surface of a body, the half-space $y > 0$, in the positive x direction. At time $t = 0$ the front is in the $x = 0$ plane. Let us assume that the body is separated into two quadrants of the space by thermal insulation along the $x = 0$ plane. Let us determine the heat propagation process in the $x > 0$, $y > 0$ domain.

In the nondimensional variables introduced in the same manner as in the preceding Section, the problem is formulated thus

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}; \qquad t = 0, \quad \theta = 0$$

$$x = 0, \quad \frac{\partial\theta}{\partial x} = 0; \qquad y = 0, \quad \frac{\partial\theta}{\partial y} = \begin{cases} -\alpha(x,t), & x < Rt \\ 0, & x > Rt \end{cases} \qquad \left(R = \frac{V}{ka^2}\right) \tag{2.1}$$

Let us note that the nondimensional parameter R is analogous to the Reynolds number in boundary layer theory since $[k] = L^{-1}$ and $[a^2] = L^2 T^{-1}$. Let us also note that the solution from [4] for a solid is the solution of boundary-layer type in precisely this parameter.

Let us construct the Green's function of the formulated problem. If the solution $U(x,y,t;\ \xi)$ satisfying the initial condition, the heat insulation condition and the condition at $y = 0$

$$\frac{\partial U}{\partial y} = \begin{cases} -1, & x < \xi,\ t > 0 \\ 0, & x > \xi,\ t > 0 \end{cases}$$

is found, then $\partial^2 U/\partial\xi\partial t$ yields the influence function of a point heat source being evolved at time $t = 0$ at the point $x = \xi$; this means the Green's function is representable as

$$G(x, y, t;\ \xi, \tau) = \frac{\partial^2 U(x, y, t-\tau;\ \xi)}{\partial\xi\partial t} \tag{2.2}$$

Let us seek the function $U(x,y,t;\ \xi)$ as a Fourier cosine integral in x

$$U(x, y, t;\ \xi) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty U(s, y, t; \xi) \cos \xi s\, ds$$

To determine $U(s,y,t;\ \xi)$ we have problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial y^2} - s^2 U; \qquad t = 0, \quad U = 0; \qquad y = 0, \quad \frac{\partial U}{\partial y} = -\sqrt{\frac{2}{\pi}} \frac{\sin \xi s}{s}$$

which can be solved by an operational method. As a result, we obtain

$$\frac{\partial U(s, y, t;\ \xi)}{\partial t} = -\sqrt{\frac{2}{\pi}} \frac{\sin \xi s}{s} \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{y^2}{4t}\right) e^{-s^2 t} \tag{2.3}$$

Differentiating (2.3) with respect to ξ and using the Poisson integral [2], we find the Green's function (2.2)

$$G(x, y, t;\ \xi, \tau) = \frac{1}{2\pi(t-\tau)} \exp\left[-\frac{y^2}{4(t-\tau)}\right]\left\{\exp\left[-\frac{(x+\xi)^2}{4(t-\tau)}\right] + \right.$$
$$\left. + \exp\left[\frac{(x-\xi)^2}{4(t-\tau)}\right]\right\} \tag{2.4}$$

Now, on the basis of the principle of addition of effects due to elementary perturbations, the solution of problem (2.1) can be written as the integral

$$\theta(x, y, t) = \int\limits_0^t \int\limits_0^{R\tau} G(x, y, t;\ \xi, \tau)\, \alpha(\xi, \tau)\, d\xi\, d\tau \tag{2.5}$$

Let us use the obtained results to estimate the accuracy of the quasi-one-dimensional solution. If, in conformity with [4], $\alpha(x,t)$ is taken as

$$\alpha(x, t) = \frac{\sqrt{R}}{\sqrt{\pi(Rt - x)}} \tag{2.6}$$

then we obtain from the quasi-one-dimensional solution ($\partial^2\theta / \partial x^2 \equiv 0$) that the surface temperature behind the shock equals one and ahead of the shock the temperature field remains unperturbed. (The presence of a temperature front is associated with the fact that the equation degenerates from an elliptic into a parabolic upon discarding $\partial^2\theta/\partial x^2$). Let us now examine what (2.5) yields for the surface temperature for a chosen $\alpha(x,t)$

$$\theta(x, 0, t) = \theta_0 = \frac{\sqrt{R}}{2\pi^{3/2}} \int\limits_0^t \int\limits_0^{R\tau} \frac{1}{t-\tau} \left\{\exp\left[-\frac{(x+\xi)^2}{4(t-\tau)}\right] + \right.$$
$$\left. + \exp\left[-\frac{(x-\xi)^2}{4(t-\tau)}\right]\right\} \frac{d\xi\, d\tau}{\sqrt{R\tau - \xi}} \tag{2.7}$$

In order to compare the result with the quasi-one-dimensional solution, it is necessary to pass to the limit as $t \to \infty$ in (2.7) and to investigate the behavior of the integral in the neighborhood of $x = Rt$. Under these conditions the first exponential in the integrand of (2.7) may be discarded. We then obtain

$$\theta(x, 0, t) = \frac{\sqrt{R}}{2\pi^{3/2}} \int_0^t \int_0^{R\tau} \frac{1}{t-\tau} \exp\left[-\frac{(x-\xi)^2}{4(t-\tau)}\right] \frac{d\xi\, d\tau}{\sqrt{R\tau - \xi}} \tag{2.8}$$

Let us make an asymptotic estimate of the integral (2.8) for large R. For $R \gg 1$ the triangular domain of integration on the (ξ, τ) plane differs slightly from the sector of a circle with center at the point (Rt, t). Taking this fact into account, let us transform to polar coordinates with center at the point (Rt, t). Let r be the distance from an arbitrary point (ξ, τ) to the center (Rt, t) and let φ/R be the polar angle measured clockwise from the line $\tau = t$. Henceforth, the components of order R^{-2} and higher will be neglected in comparison with unity. With the accuracy used, the formulas of the transformation to the new variable of intergration are

$$t - \tau = r\frac{\varphi}{R}, \qquad Rt - \xi = r \tag{2.9}$$

As $t \to \infty$, to the accuracy of quantities of order R^{-2}, we will have for the surface temperature

$$\theta_0(\lambda) = \frac{\sqrt{R}}{2\pi^{3/2}} \int_0^1 \frac{d\varphi}{\varphi\sqrt{1-\varphi}} \int_0^\infty \exp\left[-\frac{(r-\lambda)^2 R}{4\varphi r}\right] \frac{dr}{\sqrt{r}}, \qquad \lambda = Rt - x \tag{2.10}$$

The integral (2.10) is evaluated directly.

Let us integrate first with respect to r. Let us put

$$r = |\lambda| e^{-\rho}, \qquad \beta = \frac{R|\lambda|}{2\varphi}$$

Then, according to [5] (p.323), we obtain

$$\frac{\sqrt{R}}{2\pi^{3/2}} \int_0^\infty \exp\left[-\frac{(r-\lambda)^2 R}{4\varphi r}\right] \frac{dr}{\sqrt{r}} = \frac{\sqrt{|\lambda| R}}{2\pi^{3/2}} \exp\left(\frac{R\lambda}{2\varphi}\right) \int_{-\infty}^{+\infty} \exp\left[-\frac{\rho}{2} - \beta_{\cosh}\rho\right] d\rho =$$

$$= \frac{\sqrt{|\lambda| R}}{2\pi^{3/2}} \exp\left(\frac{R\lambda}{2\varphi}\right)\left(\frac{2\pi}{\beta}\right)^{1/2} e^{-\beta} = \begin{cases} \pi^{-1}\sqrt{\varphi} & (\lambda > 0) \\ \pi^{-1}\sqrt{\varphi}\exp(R\lambda/\varphi) & (\lambda < 0) \end{cases}$$

After integration with respect to φ we find $\theta_0(\lambda)$

$$\theta_0(\lambda) = 1 \quad (\lambda > 0), \qquad \theta_0(\lambda) = \operatorname{erfc}\sqrt{-\lambda R} \qquad (\lambda < 0)$$

Hence, as in the quasi-one-dimensional solution, the surface temperature behind the shock is unity. The perturbation zone ahead of the shock has an extent of order $1/R$.

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